Dirac particle in the presence of plane wave and constant magnetic fields: Path integral approach

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Abstract

The Green function (GF) related to the problem of a Dirac particle interacting with a plane wave and constant magnetic fields is calculated in the framework of path integral via Alexandrou et al. formalism according to the so-called global projection. As a tool of calculation, we introduce two identities (constraints) into this formalism, their main role is the reduction of integral's dimension and the emergence in a natural way of some classical paths, and due to the existence of constant electromagnetic field, we have used the technique of fluctuations. Hence the calculation of the (GF) is reduced to a known gaussian integral plus a contribution of the effective classical action.

1 Introduction:

The propagator of Dirac particle in an external electromagnetic field is distinguished from that of the scalar particle by a complicated spin structure. By using the known anticommuting odd Grassmann variables [1], the description of the Dirac propagator gains the possibility to acquire a representation - Path integral - similar to the case of the scalar particle modified by a spin factor (SF). This representation was discussed in various contexts. Nevertheless, the description of Dirac propagator by only bosonic variables is still unfulfilled. Berezin and Marinov [2] showed that massive particles can be described in the usual five-dimension extension. This idea was exploited by several works, among them let us quote the successful formalism of Fradkin-Gitman $[3-4]$ in the relativistic case, in which we note an important supersymmetry between the bosonic and fermionic parts [5]. This formalism saw several applications while following various computation methods $[6-10]$. Recently, a generalization of Di Vecchia and

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Ravndal's approach $[12-13]$ describing a massive Dirac particle in external vector and scalar fields, and using path integral representations according to the global and local projections, has been proposed by Alexandrou et al. [14]. This formulation is still endowed with a supersymmetric action which is derived systematically without inserting a fifth component $\psi^5(\tau)$ to the spin variable as opposed to [3].

The purpose of this paper is to develop the problem of a relativistic particle with mass m in plane wave and constant magnetic fields, and show the attractive features of spin evolution in the computation of the (SF) by using the formalism of Alexandrou et al. in its global projection (without considering the Grassmann proper time) [14] . Note that this problem has been calculated in the framework of Feynman's approach based on the T-product [15] which is not strictly a genuine path integral formulation.

In the first step, by taking into account the definition of the total potential vector which characterizes our problem, we give the general formulation of Alexandrou et al.. In the third section, we compute the (GF) by adopting the fluctuation analysis performed on both, real and Grassmann variables [9] and inserting the known identities [7] into this formulation.

Let us recall initially that the (GF) $S_g(x_b, x_a)$ given by the formalism of Alexandrou et al. in the global projection [14] is

$$S_g(x_b, x_a) = \left(i\gamma^\mu (\partial_b - gA_b)_\mu + m\right) G_g(x_b, x_a) \tag{1}$$

with

$$\begin{aligned} G_g(x_b, x_a) = {} & \frac{-i}{2k_0} \exp(i\gamma^\mu \frac{\partial}{\partial \Gamma^\mu}) \int_0^\infty dT \int DxDp \\ & \int_E \tilde{D}\psi \exp\left\{ i \int_0^T \left[-\frac{k_0}{2}\dot{x}^2 - g\dot{x}.A(x) \right.\right. \\ & + \frac{1}{2k_0}\left(p^2 - m^2\right) - i\frac{g}{k_0} F_{\mu\nu}\psi^\mu\psi^\nu \\ & \left.\left. + i\psi.\dot{\psi}\right] d\tau + \psi(0).\psi(T) \right\}\Big|_{\Gamma=0,} \end{aligned} \tag{2}$$

where g is the electron charge, $\gamma^\mu a_\nu \equiv \gamma.a$ (γ are the Dirac matrices). The x, k_0 and Γ, ψ are respectively the real and Grassmann (odd) variables. The scalar product of four-vectors, denoting by a dot, is $a.b = a^\mu b_\mu$.

The boundary conditions for bosonic variables x are

$$x(0) = x_a, \quad x(T) = x_b \tag{3}$$

and the antiperiodic boundary for the spin variables

$$E = \psi^\mu(T) + \psi^\mu(0) = \Gamma^\mu \tag{4}$$

and we have the proper normalization

$$\tilde{D}\psi = D\psi[\int_E D\psi \exp\{\psi(0).\psi(T) - \int_0^T \psi.\dot{\psi} d\tau\}]^{-1} \tag{5}$$

The effective action relative to Eq.(2) shows us the contributions of the spin degrees of freedom to the kinetic energy, as well as the contribution of coupling the photon to the electron and its spin evolution.

2 Formulation of the problem

We propose a short review of the used notations and conventions in the definition of electromagnetic field. The total potential vector $A_\mu(x)$ related to the plane wave and constant magnetic fields is a sum of two terms

$$A_\mu(x) = a_\mu\left(x^T\right) + A^p_\mu(\varphi), \tag{6}$$

where $a_\mu\left(x^T\right)$ generates the constant magnetic field B and it is a function of the transverse component of the position vector x

$$a_\mu\left(x^T\right) = \frac{1}{2} f x^T = \frac{1}{2} f_{\mu\nu}\left(x^T\right)^\nu, \tag{7}$$

with

$$f_{\mu\nu} = iB\left(\epsilon_\mu \epsilon^*_\nu - \epsilon_\nu \epsilon^*_\mu\right). \tag{8}$$

(ϵ, ϵ^*) are the basis vectors set, such as

$$\epsilon = \frac{1}{\sqrt{2}}(1, i, 0, 0), \qquad \epsilon^* = \frac{1}{\sqrt{2}}\ (1, -i, 0, 0) \tag{9}$$

satisfying

$$\epsilon.\epsilon = \epsilon^*.\epsilon^* = 0 \quad \text{and} \quad \epsilon.\epsilon^* = 1, \tag{10}$$

in which, we can define the transverse components for any vector as

$$\left(x^T\right)_\mu = \epsilon_\mu(\epsilon^*.x) + \epsilon^*_\mu(\epsilon.x). \tag{11}$$

$A^p_\mu(\varphi)$ is the transverse potential vector of plane wave, and it depends on $\varphi = k.x$.

Notice that the wave vector k has only a longitudinal component

$$k = (0, 0, -1, -1) \tag{12}$$

which implies that $k.\epsilon = k.\epsilon^* = 0$, with

$$k.A^p(\varphi) = 0 \quad \text{and} \quad k^2 = 0. \tag{13}$$

From (6), we get the total electromagnetic tensor

$$\begin{aligned} F_{\mu\nu}(\varphi) &= f_{\mu\nu} + f^p_{\mu\nu}(\varphi) \\ &= iB\left(\epsilon_\mu \epsilon^*_\nu - \epsilon_\nu \epsilon^*_\mu\right) \\ &\quad + k_\mu A'^p_\nu(\varphi) - k_\nu A'^p_\mu(\varphi), \end{aligned} \tag{14}$$

where the prime indicates the derivative with respect to φ.

By making a change on time of integration $\tau \rightarrow \frac{\tau}{k_0 e_0}$ and $T = k_0 e_0$, the action of Eq. (2) goes over to that given by the formalism of Fradkin-Gitman (in the limit χ_0(Grassmann proper time) $\rightarrow 0$) with a different sign in the (SF) and an absence of the fifth Grassmann component, hence

$$\begin{aligned} G_g\left(x_b, x_a\right) &= \frac{-i}{2} \exp(i\gamma^\mu \frac{\partial}{\partial \Gamma^\mu}) \int_0^\infty de_0 \int DxDp \int_E \tilde{D}\psi \\ & \exp\left\{ i \int_0^1 \left[-\frac{\dot{x}^2}{2e_0} - g\dot{x}.A(x) \right.\right. \\ & + \frac{e_0}{2}\left(p^2 - m^2\right) - ie_0 g F_{\mu\nu}\psi^\mu\psi^\nu \\ & \left.\left. + i\psi.\dot{\psi}\right] d\tau + \psi(0).\psi(1) \right\} \Big|_{\Gamma=0} \end{aligned} \tag{15}$$

with e_0 is a real variable.

Since the plane wave field is a function of the product $k.x$, it is preferable to introduce the two following functional identities [7]

$$\begin{aligned} &\int d\varphi_b d\varphi_a \delta\left(\varphi_a - k.x_a\right) \int D\varphi Dp_\varphi \\ &\exp\left[i \int_0^1 p_\varphi \left(\dot{\varphi} - k.\dot{x}\right) d\tau \right] = 1 \end{aligned} \tag{16}$$

and

$$\begin{aligned} &\int d\eta_b d\eta_a dp_\sigma \int D\eta Dp_\eta \\ &\exp\left\{ i \int_0^1 p_\eta \left(\dot{\eta} - k.\dot{\psi}\right) d\tau + ip_\sigma\left(\eta_a - k.\psi_a\right) \right\} = 1 \end{aligned} \tag{17}$$

into expression (15). Because of these identities, the variables φ and η are independent respectively of the scalar products $k.x$ and $k.\psi$.

The term describing the interaction between the spin and a plane wave can be written as

$$\begin{aligned} f^p_{\mu\nu}\left(\varphi\right)\psi^\mu\psi^\nu &= \left[k_\mu A'^p_\nu\left(\varphi\right) - k_\nu A'^p_\mu\left(\varphi\right)\right]\psi^\mu\psi^\nu \\ &= 2\eta\left(A'^p.\psi\right), \end{aligned} \tag{18}$$

in this case, we get

$$\begin{aligned} G_g(x_b, x_a) = &\frac{-i}{2} \exp(i\gamma^\mu \frac{\partial}{\partial \Gamma^\mu}) \int_0^\infty de_0 \int DxDp \int_E \widetilde{D}\psi \int D\eta \int Dp_\eta \int d\eta_b d\eta_a dp_\sigma \\ & \int d\varphi_b d\varphi_a \int Dp_\varphi \delta(\varphi_a - k.x_a) \exp\{ i \int_0^1 [-\frac{\dot{x}^2}{2e_0} - g\dot{x}.A(x) \\ & + \frac{e_0}{2}(p^2 - m^2) + p_\varphi(\dot{\varphi} - k.\dot{x}) - ie_0 g F_{\mu\nu}\psi^\mu\psi^\nu + i\psi.\dot{\psi} - 2ie_0 g\eta A'^p.\psi \\ & + p_\eta(\dot{\eta} - k.\dot{\psi})]d\tau + \psi(0).\psi(1) + p_\sigma(\eta_a - k.\psi_a)\}|_{\Gamma=0}, \end{aligned} \tag{19}$$

where p_σ is a odd Grassmann variable. $(\eta_b, \varphi_b, \psi_b^n)$ and $(\eta_a, \varphi_a, \psi_a^n)$ are respectively the variables $(\eta(\tau), \varphi(\tau), \psi^n(\tau))$ at $\tau = 1$ and $\tau = 0$.

3 Green Function Calculation

The genuine path integral formulation (19) contains all the dynamics of a Dirac particle moving in the combined field of a plane wave and a constant magnetic field. By considering the classical trajectories, the computation of the (GF) is reduced to the computation of the known gaussian integrals and a contribution of the effective classical action.

Let us use the transverse and longitudinal components of the vectors (x, p, ψ) as

$$x = \begin{pmatrix} x^T \\ x^L \end{pmatrix}, \; p = \begin{pmatrix} p^T \\ p^L \end{pmatrix} \quad \text{and} \quad \psi = \begin{pmatrix} \psi^T \\ \psi^L \end{pmatrix}. \tag{20}$$

Therefore, from the definitions $(8), (11)$ and (20), we get the following scalar products

$$f_{\mu\nu}\psi^\mu\psi^\nu \equiv \psi^T.\left(f\psi^T\right), \;\; A'^p.\psi \equiv A'^p.\psi^T \text{ and } k.\psi \equiv k.\psi^L. \tag{21}$$

In order to linearize the quadratic bosonic term along longitudinal plane in the integral action of Eq. (19), and make the vector p^L constant during time, we shift

$$p^L \to p^L + \frac{\dot{x}^L}{e_0} + e_0 k p_\varphi. \tag{22}$$

In other words, after considering the transverse and longitudinal components given by (20) and the shifting term (22), the successive integrations over $\left(p^L, x^L\right)$ and (p_φ, φ) then over p^T in (19), lead us to

$$\begin{aligned} G_g(x_b, x_a) = & \frac{-i}{2}\exp(i\gamma^\mu \frac{\partial}{\partial\Gamma^\mu}) \int_0^\infty de_0 \int \frac{dp^L}{(2\pi)^2} \int D\eta Dp_\eta \int_E \widetilde{D}\psi \int dp_\sigma d\eta_b d\eta_a d\varphi_b d\varphi_a \\ & \delta(\varphi_b - \varphi_a + e_0 k.p^L) \int_{x_a^T}^{x_b^T} Dx^T \delta(\varphi_a - k.x_a) \exp[ip^L.(x_b^L - x_a^L) \\ & + \frac{ie_0}{2}(p^{L2} - m^2)] \exp\{i\int_0^1 [-\frac{(\dot{x}^T)^2}{2e_0} - \frac{g}{2}x^T.f\dot{x}^T - gA^p.\dot{x}^T - ie_0 g\psi^T.f\psi^T \\ & + i\psi.\dot{\psi} - 2ie_0 g\eta A'^p.\psi^T + p_\eta(\dot{\eta} - k.\dot{\psi}^L)]d\tau + \psi(0).\psi(1) + p_\sigma(\eta_a - k.\psi_a^L)\}|_{\Gamma=0} \end{aligned} \tag{23}$$

and the extracted scalar path $\dot{\varphi} = -e_0 k.p^L$ with the evolution

$$\varphi(\tau) = -e_0 k.p^L \tau + \varphi_a. \tag{24}$$

Notice that the expression (23) is obtained after performing a transformation on vector p^T as

$$p^T \to p^T + \frac{\dot{x}^T}{e_0} + gA(x). \tag{25}$$

The part which depends on vectors $\left(p^L, x_b^L, x_a^L\right)$ in Eq. (23) corresponds to the free scalar propagator.

Now, we calculate the path integral with respect to the transverse vector x^T and $\psi(\tau)$. The presence of a constant electromagnetic field causes a particular quadraticity in the action of Eq. (23), hence it is preferable to perform a fluctuation analysis on real transverse bosonic vector x^T and on fermionic vector ψ^μ in order to extract the contribution of the fixed action (classical action) in the propagator, then

$$\begin{aligned} x^T &= X^T + Y^T &(26)\\ x^T_{b,a} &= X^T_{a,b} + Y^T_{a,b} &(27)\end{aligned}$$

and

$$\psi^\mu(\tau) = \psi_c^\mu(\tau) + \zeta^\mu(\tau), \tag{28}$$

where X^T and $\zeta^\mu(\tau)$ are respectively the real and odd Grassmann fluctuations analysis. $\psi_c^\mu(\tau)$ is fixed by the Euler-Lagrange equations (see Appendix A) and the cyclic boundary conditions on the fluctuations $\zeta^\mu(\tau)$ is chosen as

$$E_0 = \zeta^\mu(1) + \zeta^\mu(0) = 0, \tag{29}$$

then the cyclic boundary condition of classical paths is

$$\psi_c^\mu(1) + \psi_c^\mu(0) = \Gamma^\mu. \tag{30}$$

Let us consider all contributions given by Eqs. (24) and (26) – (30) for the evaluation of $G_g(x_b, x_a)$: it becomes

$$\begin{aligned} G_g(x_b,x_a) =& \frac{-i}{2}\exp(i\gamma^\mu\frac{\partial}{\partial\Gamma^\mu})\int_0^\infty de_0\int\frac{dp^L}{(2\pi)^2}\int d\varphi_b d\varphi_a d\eta_b d\eta_a\\ &\int D\eta Dp_\eta\int_{E_0}\widetilde{D}\zeta^T\widetilde{D}\zeta^L\int dp_\sigma\int_{Y_a}^{Y_b}DX^T\delta(\varphi_b-\varphi_a+e_0k.p^L)\delta(\varphi_a-k.x_a)\\ &\exp\{ip^L.(x_b^L-x_a^L)+\frac{ie_0}{2}(p^{L2}-m^2)+i\int_0^1[-\frac{(\dot{X}^T)^2}{2e_0}-\frac{g}{2}X^T.f\dot{X}^T]d\tau\\ &-i\frac{g}{2}(\int_{\varphi_a}^{\varphi_b}d\varphi A^p(\varphi)\frac{dY^T}{d\varphi}+X^T.fY^T|_{\varphi_a}^{\varphi^b})+i\int_0^1[-ie_0g\eta A'^p\psi_c^T\\ &+p_\eta\dot{\eta}-ie_0g\zeta^T.f\zeta^T+i\zeta^L.\dot{\zeta}^L+i\zeta^T.\dot{\zeta}^T]d\tau\\ &+\psi_c(0).\psi_c(1)+p_\sigma(\eta_a-k(\psi_{ca}^L+\zeta^L(0)))\}|_{\Gamma=0}, \end{aligned}\tag{31}$$

by fixing the path Y^T as

$$\left(-\frac{\dot{Y}^T}{e_0}+gfY^T-gA^p(\varphi)\right)=0, \tag{32}$$

and using

$$\int_E \tilde{D}\psi = \int_{E_0} \tilde{D}\zeta \quad \text{and} \quad \int_{x_a^T}^{x_b^T} Dx^T = \int_{X_a}^{X_b} DX^T. \tag{33}$$

The term which is a function of the real fluctuations X^T appearing in the action of expression (31) is equivalent to a known gaussian integral related to the problem of scalar particle (without (SF)) in a constant electromagnetic field [10]. In fact, by using the explicit definition of electromagnetic tensor (8), we can show that this gaussian takes a particular form in terms of the uniform constant magnetic field B and the components of X_a^T ,X_b^T (see Appendix B), where the components of X_a^T ,X_b^T in the transverse plane are defined as

$$X_a^T = \begin{pmatrix} X_a^1 \\ X_a^2 \end{pmatrix}, \; X_b^T = \begin{pmatrix} X_b^1 \\ X_b^2 \end{pmatrix}. \tag{34}$$

The (GF) should be in a symmetrical form with respect to the initial and final points in order to extract the wave functions. Therefore we symmetrize the delta function $\delta\left(\varphi_b - \varphi_a + e_0 k.p^L\right)$ by inserting its exponential form

$$\begin{aligned} &\delta\left(\varphi_b - \varphi_a + e_0 k.p^L\right) \\ &= \int dz \exp\left[iz\left(\varphi_b - \varphi_a + e_0 k.p^L\right)\right] \end{aligned} \tag{35}$$

into Eq. (31) and shifting the vector p^L to $p^L - zk$. After integrating again over z, we find

$$\begin{aligned} G_g(x_b, x_a) =& \frac{-i}{2} \exp(i\gamma^\mu \frac{\partial}{\partial\Gamma^\mu}) \int_0^\infty de_0 \int_{E_0} \widetilde{D}\zeta^T \widetilde{D}\zeta^L Dp_\eta \\ & \int D\eta \int \frac{dp^L}{(2\pi)^2} \int d\varphi_b d\varphi_a d\eta_b d\eta_a \int dp_\sigma (\frac{igB}{4\pi \sin(\frac{e_0 gB}{2})}) \\ & \delta(\varphi_a - k.x_a)\delta(\varphi_b - k.x_b) \exp\{ip^L.(x_b^L - x_a^L) + \frac{ie_0}{2}(p^{L2} - m^2) \\ & - i\frac{g}{2}(\int_{\varphi_a}^{\varphi_b} d\varphi A^p(\varphi)\frac{dY^T}{d\varphi} + X^T.fY^T|_{\varphi_a}^{\varphi^b})\} \\ & \exp\{i\frac{gB}{2}[(X_b^1 X_a^2 - X_b^2 X_a^1) - \frac{1}{2}\cot(\frac{e_0 gB}{2}) \\ & ((X_b^1 - X_a^1)^2 + X_b^2 - X_a^2)^2)] + i\int_0^1 [-ie_0 g\eta A'^p \psi_c^T + p_\eta \dot{\eta} - ie_0 g \zeta^T.f\zeta^T \\ & + i\zeta^L.\dot{\zeta}^L + i\zeta^T.\dot{\zeta}^T]d\tau + \psi_c(0).\psi_c(1) + ip_\sigma(\eta_a - k.\psi_{ca}^L) - ip_\sigma k.\zeta^L(0)\}|_{\Gamma=0}. \end{aligned} \tag{36}$$

Notice that the part which does not contain the integration over Grassmann paths in Eq. (36) describes the (GF) of a scalar particle in both plane wave and constant magnetic fields [14].

The only remaining path integral in the Eq. (36), is the (SF). With the help of the velocity variables $\omega_\mu(\tau)$ [8], we compute the gaussian integrals with respect to the ζ^T and ζ^L (see Appendix B). After integrating successively over p_η and η in Eq. (36), we deduce that the spin current projected along the wave vector k is constant during the evolution and satisfies the equation

$$\dot{\eta} = 0,\ \ \eta = \eta_a = \eta_b. \tag{37}$$

Taking into account the classical equations in appendix A, we deduce that

$$k.\dot{\psi}_c^L = 0, \quad k.\psi_c^L(1) = k.\psi_c^L(0). \tag{38}$$

The multiplication by the wave vector k on the left of the boundary condition (30), and the successive integrations over p_σ and η_a in Eq. (36), lead to

$$\eta_a = k.\psi_c^L(0) = \frac{k.\Gamma^L}{2}. \tag{39}$$

This equation preserves the induced condition by the projection of Eq. (4) along k.

It has been shown that all path integrals are reduced to the computed gaussian integrals. What remains is the contribution of the effective classical action in the calculation of the (GF). By substituting all obtained solutions in appendix A into Eq. (39), and deriving with respect to Γ, we find

$$\begin{aligned} G_g(x_b,x_a) =& \frac{-i}{2}\int_0^\infty de_0 \int \frac{dp^L}{(2\pi)^2}\int d\varphi_b d\varphi_a \Big(\frac{igB}{4\pi\sin(\frac{e_0 gB}{2})}\Big)\delta(\varphi_a - k.x_a)\delta(\varphi_b - k.x_b) \\ & \exp\{ip^L.(x_b^L - x_a^L) + \frac{ie_0}{2}(p^{L2} - m^2) - i\frac{g}{2}\Big(\int_{\varphi_a}^{\varphi_b} d\varphi A^p(\varphi)\frac{dY^T}{d\varphi} + X^T.fY^T|_{\varphi_a}^{\varphi_b}\Big) \\ & + i\frac{gB}{2}[(X_b^1X_a^2 - X_b^2X_a^1) - \frac{1}{2}\cot(\frac{e_0 gB}{2})((X_b^1 - X_a^1)^2 + (X_b^2 - X_a^2)^2)]\} \\ & \{e^{(\frac{ie_0 g}{2}B)}[1 - \gamma^\mu\gamma^\nu k_\mu\epsilon_\nu^* K(\varphi_b)]\frac{\gamma^\mu\gamma^\nu\epsilon_\mu\epsilon_\nu^*}{2}[1 + \gamma^\mu\gamma^\nu k_\mu\epsilon_\nu K^*(\varphi_a)] \\ & + e^{-(\frac{ie_0 g}{2}B)}[1 - \gamma^\mu\gamma^\nu k_\mu\epsilon_\nu K^*(\varphi_b)]\frac{\gamma^\mu\gamma^\nu\epsilon_\mu^*\epsilon_\nu}{2}[1 + \gamma^\mu\gamma^\nu k_\mu\epsilon_\nu^* K(\varphi_a)]\} \end{aligned} \tag{40}$$

with

$$\begin{aligned} K(\varphi) =& \frac{g}{2(k.p^L)}\exp\left[\frac{igB\varphi}{(k.p^L)}\right] \\ & \int_{\varphi_0}^{\varphi} d\varphi' \exp\left[\frac{igB\varphi'}{(k.p^L)}\right](\epsilon.A'^p). \end{aligned} \tag{41}$$

Here we have used the formulas

$$\begin{aligned} & 1 - \frac{1}{2}\tanh\left(\frac{\alpha}{2}\right)\gamma^\mu\gamma^\nu\left(\epsilon_\mu\epsilon_\nu^* - \epsilon_\mu^*\epsilon_\nu\right) \\ =& \left(\frac{e^{-\frac{\alpha}{2}}}{e^{-\frac{\alpha}{2}} + e^{\frac{\alpha}{2}}}\right)\gamma^\mu\gamma^\nu\epsilon_\mu\epsilon_\nu^* + \left(\frac{e^{\frac{\alpha}{2}}}{e^{-\frac{\alpha}{2}} + e^{\frac{\alpha}{2}}}\right)\gamma^\mu\gamma^\nu\epsilon_\mu^*\epsilon_\nu, \end{aligned} \tag{42}$$

and

$$\gamma^{\mu}\gamma^{\nu}\left(\frac{\epsilon_{\mu}\epsilon_{\nu}^{*}}{2}+\frac{\epsilon_{\mu}^{*}\epsilon_{\nu}}{2}\right)=1. \tag{43}$$

$K^{*}(\varphi)$ is the conjugate of $K(\varphi)$ .The final expression of $G_{g}(x_{b},x_{a})$ is quite symmetric and is identical to the one obtained in [15] .

It has been shown in [8] that the exact (SF) of a Dirac particle interacting with a plane wave field can only be described through the classical Grassmann paths due to the remarkable properties of the field, and therefore, there is no contribution of the fluctuating trajectories. Similarly, in our case, by either considering a weak magnetic field ($B \ll 1$) or neglecting the fluctuations around the classical paths in the second order variation ($\zeta^{\mu}\zeta^{\nu} \approx 0$) as a semi-classical calculation of the (SF), we find the gaussian integral given in appendix $(B-3)$ is the unity. However, the description of the spin interaction is only presented by using the classical Grassmann trajectories without performing any integration. In other words, the existence of a constant magnetic field B requires an introduction of fluctuating paths which contribute as gaussian integral to the exact computation of the (SF).

By taking the limit case of the wave vector $k \to 0$ in Eq. (40), we can deduce the influence of the constant magnetic field B on the particle, we, then, get

$$\begin{aligned} G_{g}(x_{b},x_{a}) = & \frac{-i}{2}\int_{0}^{\infty} de_{0}\int \frac{dp^{L}}{(2\pi)^{2}}\left(\frac{igB}{4\pi\sin(\frac{e_{0}gB}{2})}\right)\exp\{ip^{L}.(x_{b}^{L}-x_{a}^{L})+\frac{ie_{0}}{2}(p^{L2}-m^{2}) \\ & +i\frac{gB}{2}[(X_{b}^{1}X_{a}^{2}-X_{b}^{2}X_{a}^{1})-\frac{1}{2}\cot(\frac{e_{0}gB}{2})((X_{b}^{1}-X_{a}^{1})^{2}+(X_{b}^{2}-X_{a}^{2})^{2})]\} \\ & \{e^{(\frac{ie_{0}g}{2}B)}\frac{\gamma^{\mu}\gamma^{\nu}\epsilon_{\mu}\epsilon_{\nu}^{*}}{2}+e^{-(\frac{ie_{0}g}{2}B)}\frac{\gamma^{\mu}\gamma^{\nu}\epsilon_{\mu}^{*}\epsilon_{\nu}}{2}\} \end{aligned} \tag{44}$$

4 Conclusion

We have calculated the exact Green function (GF) for a Dirac particle interacting with a plane wave and constant transverse magnetic fields via the formalism of Alexandrou et al. (global projection). In this approach, the description of spin is established through the integration over anticommuting Grassmann trajectories. In addition, the calculation of the (GF) is based on two techniques, in the first one, we have introduced the so-called constraints (functional identities) into the formulation. These identities reduce the dimension of the integration of all projected paths along the wave vector, and extract some classical trajectories in natural way from the propagator. Indeed, it is shown that this method is very useful for the path-integral approach particularly in the presence of the plane wave field.

Due to the existence of a constant magnetic field B, we have adopted a second technique which is based on a fluctuation analysis performed on real and Grassmann variables. In fact, the paths are written in terms of both, fixed

and fluctuating trajectories, as a consequence, the path integral is reduced to a calculation of known gaussian integrals, and by inserting the classical solutions of Euler-Lagrange into the effective classical action, we obtain the exact result of the (GF) as provided in the literature.

A Classical solutions

$\psi_c^\mu(\tau)$ is fixed by Euler-Lagrange equations as

$$\dot{\psi}_c^T - e_0 g f \psi_c^T = -e_0 g \eta A'^p, \tag{A-1}$$

$$\dot{\psi}_c^L(\tau) = -\frac{i}{2} k \dot{p}_\eta. \tag{A-2}$$

The classical solutions of $\psi_c^T(\tau)$ from the Euler-Lagrange equations are

$$\psi_c^T(\tau) = -e_0 g \eta_a e^{Q\tau} \int_0^\tau \left(e^{-Q\acute{\tau}} A'^p\right) d\acute{\tau} + e^{Q\tau} \psi_c^T(0) \tag{A-3}$$

with

$$Q_{\mu\nu} = e_0 g f_{\mu\nu}. \tag{A-4}$$

We can obtain the initial and final classical solutions $\psi_c^T(0)$ and $\psi_c^T(1)$ of the spin variables from the boundary condition (30) and the general solution (A-3)

$$\begin{aligned} \psi_c^T(0) = e_0 g \eta_a e^Q \left(1 + e^Q\right)^{-1} \int_0^1 \left(e^{-Q\acute{\tau}} A'^p\right) d\acute{\tau} \\ + \frac{1}{2}\left(1 - \tanh\frac{Q}{2}\right)\Gamma^T, \end{aligned} \tag{A-5}$$

$$\begin{aligned} \psi_c^T(1) = -e_0 g \eta_a e^Q \left(1 + e^Q\right)^{-1} \int_0^1 \left(e^{-Q\acute{\tau}} A'^p\right) d\acute{\tau} \\ + e^Q \left(1 + e^Q\right)^{-1} \Gamma^T. \end{aligned} \tag{A-6}$$

In the same way as for the transverse classical solution, we have

$$\psi_c^L(\tau) = -\frac{i}{2} k p_\eta + \frac{i}{2} k p_{\eta_a} + \psi_c^L(0) \tag{A-7}$$

and

$$\psi_c^L(1) = \frac{i}{4} k \left(p_{\eta_a} - p_{\eta_b}\right) + \frac{\Gamma^L}{2}, \tag{A-8}$$

$$\psi_c^L(0) = -\frac{i}{4} k \left(p_{\eta_a} - p_{\eta_b}\right) + \frac{\Gamma^L}{2}. \tag{A-9}$$

B Gaussian integrals

The known gaussian integral that appears in the problem of scalar particle (without (SF)) in constant electromagnetic field [10] is

$$\int DX^T \exp\left[i\int_0^1 \left(-\frac{\left(\dot{X}^T\right)^2}{2e_0} - \frac{g}{2}X^T.f\dot{X}^T\right)d\tau\right]$$
$$= \frac{igB}{4\pi\sin\left(\frac{e_0gB}{2}\right)}\exp\left\{i\frac{gB}{2}\left[\left(X_b^1X_a^2 - X_b^2X_a^1\right)\right.\right.$$
$$\left.\left.-\frac{1}{2}\cot\left(\frac{e_0g}{2}B\right)\left(\left(X_b^1 - X_a^1\right)^2 + \left(X_b^2 - X_a^2\right)^2\right)\right]\right. . \tag{B-1}$$

With the help of velocity variables $\omega_\mu(\tau)$ [8], such that

$$\omega_\mu(\tau) = \dot{\zeta}_\mu(\tau),$$
$$\zeta_\mu(\tau) = \frac{1}{2}\int_0^1 \varepsilon(\tau - s)\,\omega_\mu(s)\,ds,$$
$$\varepsilon(\tau) = \text{sign of } \tau, \tag{B-2}$$

the integral of (SF) along ζ^T reduce to a simple gaussian which appears in the treatment of Dirac particle in a constant electromagnetic field

$$\int_{E_0} \tilde{D}\zeta^T \exp\left\{i\int_0^1 \left[-ie_0g\zeta^T.f\zeta^T + i\zeta^T.\dot{\zeta}^T\right]d\tau\right\}$$
$$= \left[\det\left(\cosh\left(\frac{e_0g}{2}f\right)\right)\right]^{1/2} = \cosh\left(\frac{ie_0gB}{2}\right), \tag{B-3}$$

thus the integral according to the longitudinal components

$$\int_{E_0} \tilde{D}\zeta^L \exp\left\{i\int_0^1 \left(i\zeta^L.\dot{\zeta}^L - p_\sigma k\zeta^L(0)\right)\right\} = 1, \tag{B-4}$$

since it is of type

$$\int \tilde{D}\omega^L \exp\left[\int\int \left(\omega^L(\tau)\,\varepsilon(\tau - s)\,\omega^L(s)\right.\right.$$
$$\left.\left.+I\omega^L(s)\right)d\tau ds\right] = 1, \tag{B-5}$$

with

$$I_\mu = \frac{1}{2}k_\mu p_\sigma, \tag{B-6}$$

where I_μ does not depend on time of evolution and $I^2 = 0$.

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